Existence of positive Periodic Solution for variable-coefficient Second-Order Neutral Functional Differential Equations

Research Paper

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ABSTRACT
In this paper, we considered a class of second-order neutral functional differential equations. By choosing available operators and applying Krasnoselskii’s fixed point theorem, we obtained sufficient conditions for the existence of periodic solution.

Key words: Neutral functional differential equation, positive periodic solutions, fixed-point, second order, operator.

INTRODUCTION
Two-order differential equations arise in a variety of areas in Biology, Mechanics and Economics (Hale, 1977; Kuang, 1993; Zheng, 1994). The study on neutral functional differential equations is more intricate than ordinary delay differential equations. In recent years, a series of achievements on neutral functional differential equation was reported (Evans and Ramey, 2006; Freedman and Wu, 1992; Wing-Sum et al, 2009; Li and Wang, 2005; Zhibo, 2012; Wang, 2004; Wu et al., 2008; Wu and Wang, 2007; Zhang and Wang, 2003).

Recently, Wu and Wang (2007) in their study discusses the second-order neutral delay differential equation as:

(\( x(t) - cx(t - \delta) \))\( ^00 \) + \( a(t)x(t) = \lambda b(t)f(t,x(t - \tau(t))) \)  

(1.1)

Where \( \lambda \) is a positive parameter, \( \delta \) and \( c \) are constants with \( |c| \leq 1, a(t), b(t) \in C(R, (0, \infty)), f \in C([0, \infty], [0, \infty]), \) and \( a(t), b(t), \tau(t) \) are \( \omega \)-periodic functions. The key step in the study of Wu and Wang (2007) is the application of a theorem in the study of Zhang in (1997) for the neutral operator \( (Ax)(t) = x(t) - cx(t - \delta), \) and the celebrated fixed-point index theorem, to obtain the existence of positive periodic solutions for (1.1) with \( c < 0. \)

Wu and Wang (2007) discussed the following two types of second-order neutral functional differential equations:

(\( x(t) - cx(t - \tau(t)) \))\( ^00 \) = \( a(t)x(t) - f(t,x(t - \tau(t))) \)  

(1.2)

And:

(\( x(t) - cx(t - \tau(t)) \))\( ^00 \) = \( -a(t)x(t) + f(t,x(t - \tau(t))) \)  

(1.3)

Where \( a(t) \in C(R, (0, \infty)), \tau(t) \in C(R,R), f \in C(R \times [0, \infty], [0, \infty]), \) and \( a(t), b(t), \tau(t) \) are \( \omega \)-periodic functions. By using operator equation \( Qx + sx = x \) to obtain the existence of positive periodic solutions for (1.2) and (1.3) with \( c \in [-\omega, 1). \)

Zhibo (2012) discussed the second-order neutral differential equation given as:

(\( x(t) - cx(t - \tau(t)) \))\( ^00 \) = \( -a(t)x(t) + \lambda b(t)f(x(t - \tau(t))) \)  

(1.4)

Where \( \lambda \) is a positive parameter, \( \delta \) and \( c \) are constants with \( |c| \leq 1, a(t), b(t) \in C(R, (0, \infty)), f \in C([0, \infty], [0, \infty]), \) and \( a(t), b(t), \tau(t) \) are \( \omega \)-periodic functions. Through the analysis of the generalized neutral operator and applied fixed-point index theorem, the existence of positive periodic solutions for Equation (1.4) is discussed with two cases:

\( c < 0 \) and \( c > \min\left\{ k_1, \frac{m}{M+m} \right\}; c > 0 \) and \( c < \min\left\{ \frac{m}{M+m}, \frac{LM-lm}{(L-l)M-lm} \right\}. \)

In this paper, we further discussed a neutral functional
differential equation as:

\[(x(t) - cx(t - \tau(t)))^{(0)} + p(t)x^{(0)}(t) + q(t)x(t) = f(t,x(t),x(t - \tau(t)))\]  

(1.5)

Where \(p(t) \in C(\mathbb{R},(0,\infty)), \ q(t) \in C(\mathbb{R},(0,\infty)), \tau(t) \in C(\mathbb{R},\mathbb{R}), \ f \in C(\mathbb{R} \times [0,\infty) \times [0,\infty],[0,\infty]), \ p(t), \ q(t,\tau(t), f(t,x(t),x(t - \tau(t)))\)

are \(\omega\)-periodic functions, and \(\omega, c\) are constants.

By employing Krasnoselskii’s fixed point theorem, we obtained various sufficient conditions for the existence of positive periodic solutions for Equation (1.5). Our results are more general than those in (Wing-Sum et al., 2009; Zhibo 2012; Wu and Wang, 2007) because, when \(p(t) = 0, q(t) = -a(t)\) or \(q(t) = a(t)\), then (1.5) can be transformed into (1.1) to (1.4). So, our results improved and extended the results given by these authors Wing-Sum et al. (2009), Zhibo (2012) and Wu and Wang (2007).

For the sake of convenience, we first gave some denotations and recalled a famous fixed point theorem:

Let \(X = \{x(t) \in C(\mathbb{R}) : x(t + \omega) = x(t), t \in \mathbb{R}\}\) with norm \(k_{x} = sup_{t \in [0,\omega]}|x(t)|\). Clearly, \((X,k)\) is a Banach space. Define:

\[C_{0}^{\omega} = \{x(t) \in C([0,\infty)): x(t) = x(t + \omega) = x(t)\}, \ C_{\omega}^{\omega} = \{x(t) \in C([-\infty,0]): x(t + \omega) = x(t)\}\]

Denote:

\[\mathcal{F}(t,x(t),x(t - \tau(t))) = f(t,x(t),x(t - \tau(t))) - cp(t)x^{(0)}(t - \tau(t))(1 - \tau^{(0)}(t)) - cq(t)x(t - \tau(t))\]

The following is Krasnoselskii’s fixed point theorem which our results is based on.

**Theorem A**

Let \(X\) be a Banach space. Assume \(K\) is a bounded closed convex subset of \(X\). If \(Q, S : K \rightarrow X\) satisfy (You, 1982):

1) \(Qx + S\) \(\in K\) for all \(x \in K\),
2) \(S\) is a contractive operator and
3) \(Q\) is a completely continuous operator in \(K\), then, \(Q + S\) has a fixed point in \(K\).

**Positive periodic solutions**

First, we introduced the following four Lemmas, which can be found in the study of Wang et al. (2007).

**Lemma 2.1**

Suppose that:

\[\frac{1}{Q\omega} |\exp(\int_{0}^{\omega} p(u)du) - 1| \geq 1\]  

(2.1)

Where:

\[Q = |1 + \exp(\int_{0}^{\omega} p(u)du)|^{2} R, \quad R = \max_{t \in [0,\omega]} |\frac{\exp(\int_{t}^{t+\omega} p(u)du)}{\exp(\int_{0}^{\omega} p(u)du)} - 1| q(s)ds|\]

Then, there are continuous \(\omega\)-periodic functions \(a\) and \(b\) such that \(b(t) > 0, \int_{0}^{\omega} a(u)du > 0\) and \(a(t) + b(t) = p(t), b'(t) + a(t)b(t) = q(t)\), for \(t \in \mathbb{R}\).

**Lemma 2.2**

Suppose the conditions of lemma 2.1 hold and \(\varphi \in X\), then, the equation:

\[x^{(0)} + p(t)x^{(0)} + q(t)x = \varphi(t)\]  

(2.2)

has a \(\omega\) periodic solution. Moreover, the periodic solution can be expressed by:

\[x(t) = \int_{t}^{t+\omega} G(t,s)\varphi(s)ds, \quad G(t,s) = \int_{t}^{t+\omega} \frac{\exp(\int_{t}^{s} b(v)dv) + \int_{s}^{t+\omega} \exp(\int_{s}^{t+\omega} b(v)dv + \int_{0}^{t+\omega} a(v)dv)du}{|\exp(\int_{0}^{\omega} a(u)du) - 1| |\exp(\int_{0}^{\omega} b(u)du) - 1|} \int_{0}^{\omega} a(v)dv)du\]  

(2.3)

**Lemma 2.3**

Green’s function \(G(t,s)\) satisfies the following properties:

\[G(t,t + \omega) = G(t,t), \quad G(t + \omega, s + \omega) = G(t,s)\]

\[\frac{\partial}{\partial s} G(t,s) = a(s)G(t,s) - \frac{\exp(\int_{t}^{s} b(v)dv) - 1}{\exp(\int_{0}^{\omega} b(v)dv) - 1}, \quad \frac{\partial}{\partial t} G(t,s) = -b(t)G(t,s) - \frac{\exp(\int_{t}^{s} a(v)dv) - 1}{\exp(\int_{0}^{\omega} a(v)dv) - 1}\]

**Lemma 2.4**

Let \(A = \int_{0}^{\omega} p(u)du, \quad B = \omega^{2} \exp(\frac{1}{\omega} \int_{0}^{\omega} lnq(u)du)\)\)

If:

\[A^{2} \geq 4B\]  

(2.5)

Then, we have:
\[ \min \left\{ \int_0^\omega a(u)du, \int_0^\omega b(u)du \right\} \geq \frac{1}{2}(A - \sqrt{A^2 - 4B}) := l, \]
\[ \max \left\{ \int_0^\omega a(u)du, \int_0^\omega b(u)du \right\} \leq \frac{1}{2}(A + \sqrt{A^2 - 4B}) := m. \]

Moreover:
\[ 0 < M := \frac{\omega}{(e^m - 1)^2} \leq G(t, s) \leq \frac{\omega e^p}{(e^l - 1)^2} := N, \] \tag{2.6}

And \( \sigma = \frac{M}{N} \) we know that \( 0 < \sigma < 1 \).

Next, we considered Equation (1.5) which can be rewritten as:
\[ (x(t) - cx(t - \tau(t))) = p(t)(x(t) - cx(t - \tau(t))) + q(t)x(t) - cx(t - \tau(t)) \]
\[ = f(t, x(t), x(t - \tau(t))) - cp(t)x^0(t - \tau(t))(1 - \tau^0(t)) - cq(t)x(t - \tau(t)) \] \tag{2.7}

Taking \( y(t) = x(t) - cx(t - \tau(t)) \), Equation (2.7) is transformed into:
\[ y^00(t) + p(t)y^0(t) + q(t)y(t) \]
\[ = f(t, x(t), x(t - \tau(t))) - cp(t)x^0(t - \tau(t))(1 - \tau^0(t)) - cq(t)x(t - \tau(t)) \] \tag{2.8}

Define operators \( Q, S : X \rightarrow X \) by:
\[ R(t + \omega) \]
\[ (Qx)(t) = : \quad G(t, s)[f(s, x(s), x(s - \tau(s)))] \]
\[ - cp(s)x^0(s - \tau(s))(1 - \tau^0(s)) - cq(s)x(s - \tau(s))]ds(Sx)(t) = cx(t - \tau(t)) \] \tag{2.9}

In view of Equations (2.2), (2.8), and the aforementioned analysis, the existence of periodic solutions for Equation (1.5) is equivalent to the existence of solutions for the operator equation given as:
\[ Qx + Sx = x \text{ in } X. \] \tag{2.11}

Moreover, by the complete continuity of \( P \), it is easy to verify lemma 2.5. \( Q \) is completely continuous in \( X \). Besides, we have:

**Lemma 2.6**

If \( |c| < 1 \), \( S \) is a contractive operator.

**Proof:**

For any \( x_1, x_2 \in X \), we have:
\[ |(Sx_1)(t) - (Sx_2)(t)| = |cx_1(t - \tau(t)) - cx_2(t - \tau(t))| \leq |c|kx_1 - x_2k \]

This implies that:
\[ k(Sx_1)(t) - (Sx_2)(t)k \leq |c|kx_1 - x_2k \]

Since \( |c| < 1 \), \( S \) is a contractive operator. Now we present our results on Equation (1.5).

**Theorem 2.1**

If \( c \in (0, 1) \) and
\[ \frac{1 - c}{N\omega} \leq F(t, x(t), x(t - \tau(t))) \leq \frac{1 - c}{M\omega} \]
and \( t \in \left[ \frac{c}{M}, \frac{1}{m} \right] \) \tag{2.12}

Then Equation (1.5) has at least one positive \( \omega \)-periodic solution \( x(t) \) with \( \frac{M}{N} \leq x(t) \leq \frac{N}{M} \).

**Proof:**

Let \( K_1 = \{ x \in X : x \in \left[ \frac{M}{N}, \frac{N}{M} \right] \} \). It is obvious that \( K_1 \) is a bounded closed convex set in \( X \). Moreover, for any \( x \in K_1 \), it is easy to verify that \( Q, S \) and \( FX(t, x(t), x(t - \tau(t))) \) are continuous in \( X \). Since \( F(t, x(t), x(t - \tau(t))) \geq \frac{1}{N\omega} - k_0 > 0 \), then for any \( x, y \in K_1 \), by lemma 2.4, we have:
\[ \begin{align*}
(Qu)(t) + (Sv)(t) & = f(t, s, x(s), x(s - \tau(s)) - cp(s)x^0(s - \tau(s))(1 - \tau^0(s)) - cq(s)x(s - \tau(s)))ds + cp(t - \tau(t)) \\
& \leq \frac{M}{N\omega} \cdot \frac{1}{\omega} + cM \frac{N}{M} \leq \frac{N}{M}.
\end{align*} \] \tag{2.13}

On the other hand, by lemma 2.4, we have:
\[ \begin{align*}
(Qu)(t) + (Sv)(t) & = f(t, s, x(s), x(s - \tau(s)) - cp(s)x^0(s - \tau(s))(1 - \tau^0(s)) - cq(s)x(s - \tau(s)))ds + cp(t - \tau(t)) \\
& \geq M \frac{N}{M} \cdot \frac{1}{\omega} + cM \frac{N}{M} = \frac{M}{N}.
\end{align*} \tag{2.14}

Combining Equations (2.13) and (2.14), we get \( Qx + Sy \in K_1 \) for all \( x, y \in K_1 \).

Moreover, from lemma 2.5 and 2.6, \( Q \) is completely continuous and \( S \) is a contractive operator in \( X \). Hence, by Theorem A, \( Q + S \) has a fixed point \( x \in K_0 \), that is to say,
Equation (1.5) has a positive \( \omega \)-periodic solution \( x(t) \) with \( \frac{N}{M} \leq x(t) \leq \frac{N}{M} \).

**Theorem 2.2**

If \( c = 0 \) and \( \frac{1}{N \omega} < F(t, x(t), x(t - \tau(t))) \leq \frac{1}{M \omega} \), then, Equation (1.5) has at least one positive \( \omega \)-periodic solution \( x(t) \) with \( \frac{N}{M} \leq x(t) \leq \frac{N}{M} \).

**Proof:**

Let \( Q, S \) be defined as in (2.15). In this case, especially, \( S \) is degenerated into a zero operator. Let \( K_1 = \{ x \in X : x \in [\frac{N}{M}, \frac{N}{M}] \} \) as in Theorem 2.1. Following the steps in the proof of Theorem 2.1, we easily get that the equation (1.5) has at least one positive \( \omega \)-periodic solution \( x(t) \) with \( \frac{N}{M} \leq x(t) \leq \frac{N}{M} \).

**Theorem 2.3**

Assume that \( c \in (-\frac{M}{N}, 0) \) and if:

\[-c < F(t, x(t), x(t - \tau(t))) \leq \frac{M}{N} \quad \text{for all} \quad t \in [0, \omega] \quad \text{and} \quad x \in [0, M \omega],\]

Then, Equation (1.5) has at least one positive \( \omega \)-periodic solution \( x(t) \) with \( 0 < x(t) \leq M \omega \).

**Proof:**

Define \( Q, S \) as in (2.9)-(2.10) and \( K_2 = \{ x \in X : x \in [0, M \omega] \} \). Obviously, \( K_2 \) is a bounded closed convex set and \( Q(K_2) \subset X \), \( S(K_2) \subset X \). Lemmas 2.5 and 2.6 still hold, and we must have \( Qx + Sy \in K_2 \) for all \( x, y \in K_2 \).

In fact, for any \( x, y \in K_2 \), we have from Lemma 2.4 that:

\[
(Qx)(t) + (Sy)(t) = \int_{t_0}^{t} G(t, s)[f(s, x(s), x(s - \tau(s))) - c_0(s)x(s - \tau(s))(1 - r(s)) - c_0(s)x(s - \tau(s))]ds + c_0(t - \tau(t)) \\
\leq \frac{M}{N} \int_{t_0}^{t} [f(s, x(s), x(s - \tau(s))) - c_0(s)x(s - \tau(s))(1 - r(s)) - c_0(s)x(s - \tau(s))]ds + c_0(t - \tau(t)) \\
\leq \frac{N}{M} \cdot \frac{M}{N} \\
\leq \frac{M}{N} \cdot \frac{M}{N}.
\]

(2.15)

On the other hand, by lemma 2.4, we have:

\[
(Qx)(t) + (Sy)(t) = R \int_{t_0}^{t} G(t, s)[f(s, x(s), x(s - \tau(s))) - c_0(s)x(s - \tau(s))(1 - r(0)) - c_0(s)x(s - \tau(s))]ds + c_0(t - \tau(t)) \geq M R \int_{t_0}^{t} [f(s, x(s), x(s - \tau(s))) - c_0(s)x(s - \tau(s))(1 - r(0)) - c_0(s)x(s - \tau(s))]ds + c_0(t - \tau(t)) \geq M \omega (-c) + cM \omega = 0.
\]

(2.16)

So, from Equations (2.15) and (2.16), we get \( Qx + Sy \in K_2 \) for all \( x, y \in K_2 \). Since \( F(t, x(t), x(t - \tau(t))) > -c \), from Equations (2.11) and (2.16), it is clear that \( x(t) > 0 \), hence, \( 0 < x(t) \leq M \omega \), that is to say the Equation (1.5) has at least a positive \( \omega \)-periodic solution \( x(t) \) with \( 0 < x(t) \leq M \omega \).

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